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Correlators of $N = 1$ superconformal currents

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Abstract
We give an explicit expression for the M-point correlator of the superconformal current in two-dimensional $N = 1$ superconformal field theories.

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1. Main result

$N = 1$ superconformal field theories [1] have a chiral algebra generated by a spin-2 field T known as the stress energy tensor and a spin-3/2 field G known as the superconformal current, which is the superpartner of T. The singular terms of the operator product expansion of these fields are given by

$$T(z)T(z') = \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial T(z')}{z-z'} + \mathcal{O}(1) \quad (1)$$

$$T(z)G(z') = \frac{(3/2)G(z')}{(z-z')^2} + \frac{\partial G(z')}{z-z'} + \mathcal{O}(1) \quad (2)$$

$$G(z)G(z') = \frac{(2c/3)}{(z-z')^3} + \frac{2T(z')}{z-z'} + \mathcal{O}(1), \quad (3)$$

where c is the central charge. The form written here is generic so long as there are no further conserved currents [2].

The point of this paper is to give an expression for the correlator

$$C_M = \langle G(z_1)G(z_2)\ldots G(z_M)\rangle \quad (4)$$

where M must be even or the correlator will vanish. Obviously the two-point correlator is directly given by the OPE as

$$C_2 = \frac{(2c/3)}{(z_1-z_2)^3}. \quad (5)$$

Using equations (2) and (3) we can examine the pole structure with respect to one variable z_1, and derive a recursion relation for this correlator (compare, for example, with the discussions in [2, 3]) whereby the M point correlator is written in terms of the $M-2$ point correlator

$$C_M = \sum_{i=2}^{M} (-1)^i \left(\underbrace{\frac{(2c/3)}{(z_1 - z_i)^3}}_{\alpha} + \frac{2}{(z_1 - z_i)} \sum_{j \neq 1,i} \left[\underbrace{\frac{3/2}{(z_i - z_j)^2}}_{\beta} + \underbrace{\frac{1}{z_i - z_j} \frac{\partial}{\partial z_j}}_{\gamma} \right] \right) C_{M-2}(\hat{1}, \hat{\imath}) \tag{6}$$

where $C_{M-2}(\hat{1}, \hat{\imath})$ is the $M-2$ point correlator that excludes coordinates z_1 and z_i

$$C_{M-2}(\hat{1}, \hat{\imath}) = \langle G(z_2) \ldots G(z_{i-1}) G(z_{i+1}) \ldots G(z_M) \rangle. \tag{7}$$

The terms in equation (6) are labeled α, β, γ for future reference. Thus through this recursion relation (and starting the recursion with equation (5)) in principle the M point correlator can be determined.

We claim that the solution to this recursion relation can be written in a very simple closed form. We write the correlator in terms of a fully analytic 'wavefunction [4]' ψ_M

$$C_M = \psi_M V_M^{-3} \tag{8}$$

where

$$V_M = \prod_{1 \leqslant i < j \leqslant M} (z_i - z_j). \tag{9}$$

Inspired by the construction of [5], we write the wavefunction as

$$\psi_M = \mathcal{N} \sum_{P \in S_M} \prod_{1 < r < s \leqslant M/2} \chi(z_{P(2r-1)}, z_{P(2r)}; z_{P(2s-1)}, z_{P(2s)}) \tag{10}$$

with the normalization constant

$$\mathcal{N} = \frac{(c/3)^{(M/4)(3-M/2)}}{(M/2)!}. \tag{11}$$

Here, P is a permutation of the integers $1, \ldots, M$ and the sum is over all such permutations. The function χ is given by

$$\begin{aligned} \chi(z_1, z_2; z_3, z_4) = {} & A(z_1 - z_3)^3 (z_2 - z_4)^3 (z_1 - z_4)^3 (z_2 - z_3)^3 \\ & + (z_1 - z_3)^4 (z_2 - z_4)^4 (z_1 - z_4)^2 (z_2 - z_3)^2, \end{aligned} \tag{12}$$

where $A = (c/3) - 1$. Perhaps a more useful form is given by merging V with ψ to write directly

$$\begin{aligned} C_M = \frac{1}{(M/2)!} \sum_{P \in S_M} \sigma(P) \Bigg[& \prod_{1 < r < s \leqslant M/2} \tilde{\chi}(z_{P(2r-1)}, z_{P(2r)}; z_{P(2s-1)}, z_{P(2s)}) \\ & \times \prod_{1 < t \leqslant M/2} \frac{(c/3)}{(z_{P(2t-1)} - z_{P(2t)})^3} \Bigg] \end{aligned} \tag{13}$$

where $\sigma(P)$ is the signature (± 1) of the permutation and where $\tilde{\chi}$ is given by

$$\begin{aligned} \tilde{\chi}(z_1, z_2; z_3, z_4) &= \frac{3}{c} \left[A + \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \right] \\ &= 1 + (3/c) \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \end{aligned} \tag{14}$$

Using Mathematica™, we have been able to verify that this form of the correlator satisfies the recursion relation for $M \leqslant 10$. Below we sketch a proof that this recursion relation holds for general M.

We note in passing that correlators of spin-1 and spin-2 fields of chiral algebras have been found to have somewhat similar (but in some ways even simpler) forms. See the discussions in [6] of the work of [7].

2. Special cases and further checks

There are several special cases where C_M may be calculated or tested by other means to check the validity of our result. These cases are $c = 1, 3/2, -6/5, 7/10, -21/4$ and ∞.

Free Boson. The $c = 1$ case is a free boson theory where G is given by

$$G \sim [: \mathrm{e}^{\mathrm{i}\sqrt{3}\phi(z)} : - : \mathrm{e}^{-\mathrm{i}\sqrt{3}\phi(z)} :], \tag{15}$$

where ϕ is a free boson (see, for example, [8]). To evaluate the correlator of many G fields, we multiply out the terms in all combinations. The only terms that can be nonzero, must have overall charge neutrality [1] so that the number of terms $\mathrm{e}^{\mathrm{i}\sqrt{3}\phi}$ must be the same as the number of terms of $\mathrm{e}^{-\mathrm{i}\sqrt{3}\phi}$. Thus all terms are of the form

$$\prod_{i,j\in A; i<j} (z_i - z_j)^3 \prod_{i,j\in B; i<j} (z_i - z_j)^3 \prod_{i\in A, j\in B} (z_i - z_j)^{-3} \tag{16}$$

where all of the particles are divided up into two groups A and B of equal size (compare this form to that used in [9]). Summing all such terms should be equivalent to our predicted form of C_M for $c = 1$. This has been verified numerically for $M \leqslant 12$.

Ising Cubed. The $c = 3/2$ case can be thought of as the cube of the Ising CFT, the Virasoro minimal model $M(3, 4)$, which has $c = 1/2$. G is simply the product of the three Majorana fields, one from each theory. The correlator of Majorana fields in $M(3, 4)$ is a simple Pfaffian [1] written as $\mathrm{Pf}[1/(z_i - z_j)]$ thus the correlator of the G fields should be the cube of this Pfaffian. Again we can verify numerically for $M \leqslant 12$ particles that our predicted form for C_M is indeed the cube of this Pfaffian for $c = 3/2$.

$M(3, 5)$ *Squared.* The $c = -6/5$ case can be thought of as the square of the Virasoro minimal model $M(3, 5)$ which has $c = -3/5$. Each $M(3, 5)$ theory contains a field with spin $3/4$, and the field G is then the product of these two fields. The correlator of the fields with weight $3/4$ in the $M(3, 5)$ theory has been given explicitly in [10], and the correlator of the G fields is then the square of this quantity. Again we verify numerically for $M \leqslant 12$ particles that our predicted form for C_M with $c = -6/5$ agrees with this quantity.

Tricritical Ising. The case of $c = 7/10$ is the tricritical Ising [1] model, the Virasoro minimal model $M(4, 5)$ where G corresponds to the field $\phi_{3,1}$. The four-point correlator of this field has been calculated in [11] and indeed agrees with our expression. Generally, the many-point correlator of this field must satisfy a particular differential equation corresponding to a null vector condition (see equation (D8) from BPZ [12]). One can check that our proposed form for C_M with $c = 7/10$ does indeed satisfy this differential equation for small M. This has been verified for $M \leqslant 8$ particles using Mathematica™.

$M(3, 8)$. The Virasoro minimal model $M(3, 8)$ is the $c = -21/4$ case where G corresponds to the field $\phi_{2,1}$. This field must similarly satisfy a differential equation corresponding to a null vector condition (equation (5.17) of BPZ [12]). One can again check that our proposed form for C_M satisfies this condition for $M \leqslant 8$ using Mathematica™.

'Semiclassical' Limit: The limit of $c = \infty$ is trivial to solve. In this limit, our proposed expression for the correlator, equation (13), takes the form of a Pfaffian $\mathrm{Pf}[1/(z_i - z_j)^3]$. In the $c = \infty$ limit only the α term survives in the recursion relation equation (6). It is then easy to see that this proposed Pfaffian form does indeed satisfy this simplified recursion relation.

3. Sketch of proof

In this section we prove that the recursion relation equation (6) holds for general M (for small M it is easy to check explicitly). The proof is messy but shows how, with sufficient algebra, the recursion relation can be established. It seems likely a more elegant proof may also be found.

To check the recursion relation, in general, we examine the poles that occur when z_1 approaches some other particle z_i as in equation (6). For simplicity we consider the case of $i = 2$.

Examining the form of equations (13) and (14), it is clear that the only way to obtain a third order pole when 2 approaches 1 is when the permutation pairs particles 1 and 2 so that $P(2r) = 1$ and $P(2r - 1) = 2$ or vice versa. Further, from equation (14) it is clear that in the product of $\tilde{\chi}(1, 2; m, n)$ terms only the '1' term in equation (14) can contribute from $\tilde{\chi}$ or the pole will be lower order. Counting the number of permutations that contribute, we see that the coefficient of the third order pole is $C_{M-2}(\hat{1}, \hat{2})$ with precisely the right coefficients to account for the α term of equation (6).

We must next show that there is no second order pole as z_2 approaches z_1 (since there is no second order pole in equation (6)). Again, using the form of equations (13) and (14) it is clear that the only way to obtain a second order pole is again when the permutation pairs particles 1 and 2 so that $P(2r) = 1$ and $P(2r - 1) = 2$ or vice versa, and then when we multiply out the product of the $\tilde{\chi}$ terms we should include exactly one factor of $(z_1 - z_2)(z_m - z_n)/(z_1 - z_n)(z_2 - z_m)$. Keeping such terms then taking the limit of $z_1 \to z_2$ we see that the resulting expression becomes symmetric under interchange of 1 and 2 to leading order. However the sum in equation (13) is antisymmetrized because of the factor of $\sigma(P)$ thus canceling this term.

Finally we turn to the single pole as z_1 approaches z_2. There are three ways to get a first order pole. In the first two ways, 1 and 2 are in the same pair, and terms in the product of $\tilde{\chi}$'s give $(z_1 - z_2)^2$ which then combines with the $1/(z_1 - z_2)^3$ term in equation (13) to give a single pole. One way to get such a quadratic term is by expanding terms of the form (here we use the shorthand notation $(12) = z_1 - z_2$)

$$\lim_{1\to 2} \tilde{\chi}(1, 2; 3, 4) = 1 + (3/c)\frac{(12)(34)}{(24)(23)} - (3/c)\frac{(12)^2(34)}{(24)^2(23)} + \cdots \qquad (17)$$

to second order as shown. A second way is via the product of two $\tilde{\chi}$'s

$$\lim_{1\to 2} \tilde{\chi}(1, 2; 3, 4)\tilde{\chi}(1, 2; 5, 6) = \cdots + (3/c)^2\frac{(12)^2(34)(56)}{(23)(24)(25)(26)} + \cdots. \qquad (18)$$

The third way to get a first order pole is when particles 1 and 2 are not paired, in which case we have terms of the type

$$\lim_{1\to 2} \tilde{\chi}(1, 4; 3, 2) = 1 + (3/c)\frac{(24)(32)}{(12)(43)} + \cdots. \qquad (19)$$

Note that in all three equations we have replaced 1 with 2 in all places (since we want the evaluation at the pole) except to make the (12) factors explicit. The sum of the coefficients of these residues should give the β and γ terms of the recursion relation equation (6).

Once we have isolated the residues of these poles, we would like to find the further poles with respect to the approach of a third particle toward the second (the third particle is j in equation (6)). In equation (17) there is a second order pole for $j = 4$. Then taking position 4 to approach 2, the coefficient of this pole is then just the product of all $\tilde{\chi}$ that are not a function of positions 1 or 2 (we use only the '1' term from any other $\tilde{\chi}(1, 2; n, m)$ so we do not pick up any more powers of (12)) times the factors of $1/(z - z)^3$ not including $1/(12)^3$. Summing over these obviously gives $C_{M-2}(\hat{1}, \hat{2})$ however the coefficient is only 1/3 that necessary to account for the coefficient of term β. Let us call this contribution term A.

To evaluate the first order pole as 2 approaches 4, we expand the numerator of equation (17) so that (34) = (32) + (24). Having first taken the (12) residue, and the first order pole as 4 approaches 2 then gives $-1/(43)$ times all terms not including 1 and 2. There is obviously another way to get a first order pole from equation (17) by taking $j = 3$ (i.e., when 3 approaches 2), in this case the residue is exactly the same as the previous case. Let us call these contributions terms B.

We now turn to the poles that arise from the form of equation (18). Here, there is no possibility of getting a second order pole as any particle (excluding 1) approaches 2. The single poles are, for example, when 2 approaches 4 we have a residue $-(56)/(45)(46)$ times factors of $\tilde{\chi}$ that do not contain 1 and 2 and times the $1/(z - z)^3$ factors not including $1/(12)^3$. Call these terms C. Note that as compared to term B, these terms contain one additional power of $3/c$.

We now examine terms of the type of equation (19) which are the most complicated. As written there, the relevant permutation has $P(1) = 1$, $P(2) = 4$, $P(3) = 3$, $P(4) = 2$ and let us assume $P(i) = i$ for $i > 4$ for simplicity here. The relevant term of the permutation sum is of the form

$$\frac{(c/3)^2\tilde{\chi}(1,4,3,2)}{(z_1-z_4)^3(z_3-z_2)^3}\left[\prod_{n>2}\tilde{\chi}(1,4;2n-1,2n)\right]\left[\prod_{n>2}\tilde{\chi}(3,2;2n-1,2n)\right]$$
$$\times\left[\prod_{n>m>2}\tilde{\chi}(2m-1,2m;2n-1,2n)\right]\left[\prod_{n>2}(c/3)(z_{2n-1}-z_{2n})^{-3}\right]. \tag{20}$$

As 2 approaches 1 there is a single pole, as expressed in equation (19). Other than this pole, any occurrence of 1 in this equation is then replaced by 2 (since we are looking at the residue of the pole). Then due to the factor of $1/(24)^3$ out front there will be a second order pole as 4 approaches 2 with a prefactor of $-(c/3)/(43)(32)^2$ times the terms in brackets. Considering the first term in brackets 1 is replaced by 2, and in multiplying out these factors of $\tilde{\chi}$ we must only keep the '1' term since the other terms have a factor of (24) and reduce the order of the (24) pole. The coefficient of the second order pole as 2 approaches 4 then is given by the remaining factors with 2 replaced by 4 everywhere it occurs. This then results in precisely the same quantity (and the same sign once we include the $\sigma(P)$ factor) as in term A above. Further there will be an identical term from exchanging coordinates 3 and 4, leaving 1 and 2 alone, and exchanging each additional coordinate with its partner ($P(2n) \leftrightarrow P(2n - 1)$). Adding these two terms with term A above gives precisely the right coefficient to account for term β in equation (6).

In terms of the type of equation (19) considering the residue evaluated when 1 approaches 2, there is also the possibility of a first order pole as particle 2 approaches another coordinate. There are two ways this may occur. The first way is similar to the above paragraph, only here we take a subleading term, expanding 2 around the position of 4 (i.e, we consider the second order pole $1/(24)^2$ as in the previous paragraph, but we expand coordinate 2 around position 4 so that at first order there is also a factor of (24) in the numerator). The coefficient of the first order pole of 2 with respect to 4 is then

$$\frac{\partial}{\partial z_2}\bigg|_{z_2\to z_4}\left\{\frac{(32)}{(43)}\frac{(c/3)}{(32)^3}\left[\prod_{n>2}\tilde{\chi}(2,4;2n-1,2n)\right][\,][\,][\,]\right\} \tag{21}$$

where the three empty brackets are the same as the last three brackets in equation (20). Let us first see what happens when the derivative acts on the outside factor of (32). The derivative is a trivial -1, and we simply replace 2 by 4 everywhere it occurs in the remaining expression. Note that in the first bracketed expression, when $2 \to 4$ only the '1' term survives. The remaining expression looks just like the terms that would contribute to $C_{M-2}(\hat{1},\hat{2})$ except for the $1/(43)$ out front. We identify these terms as being exactly the B terms discussed above with a sign that precisely cancels them. Similarly, let us examine what happens when the derivative acts on the first bracketed quantity. Here, we get terms of the form $(2n-1,2n)/(4,2n-1)(4,2n)$ times something that looks exactly like our previous expression without the $1/(43)$ and with one additional factor of $3/c$. We similarly identify these terms as being exactly the C terms discussed above with a sign to cancel them. We now let the derivative act on the remaining terms in equation (21). Once we take 2 to 4, and sum over all terms we realize that this is precisely $\partial_{z_4}C_{M-2}(\hat{1},\hat{2})$ which correctly gives us the final γ term of equation (6).

To complete the proof, we realize (as mentioned above) that there is one more possible source of first order poles as particle 2 approaches other particles in terms of the type of equation (19) (here we have already taken the residue of the (12) pole and we are considering 2 approaching particles other than 1 or 4). Here we note that such terms all sum to zero. The terms in question are of the type $\tilde{\chi}(1,4;3,2)\tilde{\chi}(3,2;5,6)\tilde{\chi}(1,4;5,6)$. From the first term there is a single pole as 2 approaches 1, then from the second term there is then a single pole as, say, 5 approaches 2. However, it is easy to show that this combination will be precisely canceled by the same poles that occur in the term of the form $\tilde{\chi}(1,6;3,2)\tilde{\chi}(3,2,5,4)\tilde{\chi}(1,6,5,4)$ (with all remaining coordinates unchanged).

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